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On Symplectic and Multisymplectic Structures and Their Discrete Versions in Lagrangian Formalism

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Abstract *We introduce the Euler–Lagrange cohomology to study the symplectic and multisymplectic structures and their preserving properties in finite and infinite dimensional Lagrangian systems respectively. We also explore their certain difference discrete counterparts in the relevant regularly discretized finite and infinite dimensional Lagrangian systems by means of the difference discrete variational principle with the difference being regarded as an entire geometric object and the noncommutative differential calculus on regular lattice. In order to show that in all these cases the symplectic and multisymplectic preserving properties do not necessarily depend on the relevant Euler–Lagrange equations, the Euler–Lagrange cohomological concepts and content in the configuration space are employed.*

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1 Introduction

It is well known that the symplectic and multisymplectic structures play crucially important roles in the symplectic and multisymplectic algorithms for the finite dimensional Hamiltonian systems[1,2] and infinite-dimensional Hamiltonian systems respectively. These algorithms are very powerful and successful in numerical calculations of the relevant systems in comparison with other various non-symplectic computational schemes since the symplectic and multisymplectic schemes preserve the symplectic structure and multisymplectic structure of the systems in certain sense.

In this paper, in a simple and direct manner, we present the symplectic structure in the Lagrangian mechanism and the multisymplectic structure in the Lagrangian field theory and their preserving properties as well as the difference-type discrete versions of these issues. We employ the ordinary exterior differential calculus in the configuration space and introduce what is named the Euler–Lagrange (EL) cohomology to show that the symplectic and multisymplectic structures are preserved without necessarily making use of the EL equations in general. And the EL equations are derived from the variational principle of the relevant action functionals. Therefore, it is important to emphasize that these structure-preserving properties are established in the function space with the EL cohomology on the configuration space in general rather than in the solution space of the EL equations only.

One of the key points different from the other approaches in our approach is the EL cohomology we will introduced. Some EL cohomological concepts and content such as the EL one-forms, the coboundary EL one-forms and the EL conditions, i.e., the EL one-forms are closed,

null EL one-form gives rise to the EL equations. Although the null EL one-form is a special case of the coboundary EL one-forms which are cohomologically trivial and automatically satisfy the (closed) EL condition, they are ***not*** the same in principle. As a matter of fact, the EL cohomology is nontrivial in general. This point plays a crucial role in our approach.

In the course of numerical calculation, the "time" $t \in R$ is always discretized, say, with equal spacing $h = \Delta t$ and the space coordinates are also discretized in various cases, especially, for the Lagrangian field theory. In addition to these computational approach, there also exist various discrete physical systems with discrete or difference discrete Lagrangian functions. It is well known that the differences of functions do not obey the Leibniz law. In order to explore the discrete symplectic and multisymplectic structures in these difference discrete systems and their preserving properties in certain difference discrete versions, some noncommutative differential calculus (NCDC) should be employed[5−7] even for the well-established symplectic algorithms. This is the second key point of this paper.

Another key point of this paper different from others is that the difference discrete variational principle (DDVP) will be employed in this paper. In view of NCDC, a forward or backward difference as the forward or backward discrete derivative should be regarded as an entire geometric object respectively. In the DDVP with forward (or backward) difference, we prefer to adopt this point of view. We also show that DDVP leads to the correct results in the sense that the results not only correspond to the correct ones in the continuous limit but also are the

The plan of this paper is as follows. We first briefly rederive some well-known contents on symplectic and multisymplectic structures and their preserving properties in the Lagrangian formalism for the finite and infinite dimensional systems respectively in Sec. 2. The important issues of this section is to introduce the EL cohomology, including some cohomological concepts and content such as the EL one-forms, the coboundary EL one-forms and the EL conditions and to show that it is nontrivial in each case. In order to explain those symplectic and multisymplectic geometry and relevant preserving properties in relevant systems the EL cohomology plays a very important role. In Sec. 3, we first explain the DDVP in our approach to give rise to the discrete Euler–Lagrange (DEL) equations. Then we study the difference discrete versions of the cohomological concepts and content as well as the symplectic and multisymplectic structures in Lagrangian formalism given in Sec. 2. We present some remarks in Sec. 4. Finally, in the Appendix, some relevant NCDC on regular lattice with equal spacing on each direction is given.

2 The Symplectic and Multisymplectic Structures in Lagrangian Mechanism and Field Theory

In this section, we recall some well-known contents on symplectic and multisymplectic structures and their preserving in the Lagrangian formalism for the finite and infinite dimensional systems respectively. The important point is to introduce the EL cohomological concepts and content related to the EL equations and explain their important roles in the symplectic and multisymplectic geometry in these systems respectively.

2.1 *The Symplectic Structure in Lagrangian Mechanism*

We begin with the Lagrangian mechanism. Let time $t \in R^1$ be the base manifold, M the configuration space on t with coordinates $q^i(t)$, $i = 1, \ldots, n$, TM the tangent bundle of M with coordinates $q^i, \dot{q}^j$, $F(TM)$ the function space on TM.

The Lagrangian of the systems is denoted as $L(q^i, \dot{q}^j)$, where $\dot{q}^j$ is the derivative of q^j with respect to t. The variational principle gives rise to the well-known EL equations

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = 0 \,. \tag{1}$$

Let us take the exterior derivative d of the Lagrangian function, we get

$$\mathrm{d}L = \Big\{\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\Big\}\mathrm{d}q^i + \frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i\Big\} \,.$$

Defining the EL one-forms on T^*M,

$$E(q^i, \dot{q}^i) := \Big\{\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\Big\}\mathrm{d}q^i \,, \tag{2}$$

we have

$$\mathrm{d}L(q^i, \dot{q}^i) = E(q^i, \dot{q}^i) + \frac{\mathrm{d}}{\mathrm{d}t}\theta \,, \tag{3}$$

where θ is the canonical one-form

$$\theta = \frac{\partial L}{\partial \dot{q}^i}\mathrm{d}q^i \,. \tag{4}$$

It is easy to see from the definition (2) and Eq. (3) that first the null EL one-form is corresponding to the EL equation, secondly the null EL one-form is a special case of the coboundary EL one-forms,

$$E(q^i, \dot{q}^j) = \mathrm{d}\alpha(q^i, \dot{q}^j) \,, \tag{5}$$

where $\alpha(q^i, \dot{q}^j)$ is an arbitrary smooth function of $(q^i, \dot{q}^j)$, and thirdly the EL one-forms are not coboundary in general since θ is not a coboundary so that the EL cohomology is not trivial.

Making use of nilpotency of d on T^*M, $\mathrm{d}^2 L(q, \dot{q}) = 0$, it is easy to show that if and only if the EL one-form is closed with respect to d, which may be named the EL condition, i.e.,

$$\mathrm{d}E(q^i, \dot{q}^j) = 0 \,, \tag{6}$$

the symplectic conservation law with respect to t follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega = 0 \,, \tag{7}$$

where the symplectic structure ω is given by

$$\omega = \mathrm{d}\theta = \frac{\partial^2 L}{\partial \dot{q}^i \partial q^j}\mathrm{d}q^i \wedge \mathrm{d}q^j + \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\mathrm{d}q^i \wedge \mathrm{d}\dot{q}^j \,. \tag{8}$$

It is important to note that although the null EL one-form and the coboundary EL one-forms satisfy the EL condition, it does not mean that the closed EL one-forms can always be exact. Namely, as mentioned above, the EL cohomology is not trivial so that the EL condition is ***not*** cohomologically equivalent to the null EL one-form or the coboundary EL one-forms in general. This also means that $q^i(t)$, $i = 1, \ldots, n$ in the EL condition are ***not*** in the solution space of the EL equations only. In fact, they are still in the function space in general. Therefore, the symplectic two-form ω is conserved not only in the solution space of the equations but also in the function space in general with respect to the duration of t if and only if the EL condition is satisfied.

In order to transfer to the Hamiltonian formalism, we introduce conjugate momentum

$$p_j = \frac{\partial L}{\partial \dot{q}^j} \,, \tag{9}$$

and take a Legendre transformation to get the Hamiltonian function

$$H(q^i, p_j) = p_k \dot{q}^k - L(q^i, \dot{q}^j) \tag{10}$$

Then the EL equations become the canonical equations as follows:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}^j = -\frac{\partial H}{\partial q^j}\,. \tag{11}$$

It is clear that a pair of the EL one-forms should be introduced now

$$E_1(q^i, p_j) = \Big(\dot{q}^j - \frac{\partial H}{\partial p_j}\Big)\mathrm{d}p_j\,,$$
$$E_2(q^i, p_j) = \Big(\dot{p}_j + \frac{\partial H}{\partial q^j}\Big)\mathrm{d}q^j\,. \tag{12}$$

In terms of $z^T = (q^i, \ldots, q^n, p_1, \ldots, p_n)$, the canonical equations and the EL one-form become

$$\dot{z} = J^{-1}\nabla_z H\,, \tag{13}$$

$$E(z) = \mathrm{d}z^T(Jz - \nabla_z H)\,. \tag{14}$$

Now it is straightforward to show that the symplectic structure preserving law

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega = 0, \qquad \omega = \mathrm{d}z^T \wedge J\mathrm{d}z \tag{15}$$

holds *if and only if* the (closed) EL condition is satisfied

$$\mathrm{d}E(z) = 0\,. \tag{16}$$

2.2 *The Multisymplectic Structure in Lagrangian Field Theory*

We now consider the multisymplectic structure in Lagrangian field theory for the scalar fields. Let X be an n-dimensional base manifold with coordinates x^μ, $\mu = 1, \ldots, n$, M the configuration space on X with scalar field variables $u^i(x)$, $i = 1, \ldots, s$, TM the tangent bundle of M with coordinates u^i, u^j_μ, $u^j_\mu = \partial u^i/\partial x^\mu$, $F(TM)$ the function space on TM.

The Lagrangian of the systems under consideration is $L(u^i, u^j_\mu)$ with the well-known EL equations from the variational principle,

$$\frac{\partial L}{\partial u^i} - \frac{\partial}{\partial x^\mu}\frac{\partial L}{\partial u^i_\mu} = 0\,. \tag{17}$$

Let us introduce the EL one-form in the function space $F(TM)$,

$$E(u^i, u^j_\mu) := \Big\{\frac{\partial L}{\partial u^i} - \frac{\partial}{\partial x^\mu}\frac{\partial L}{\partial u^i_\mu}\Big\}\mathrm{d}u^i\,. \tag{18}$$

It is easy to see that the null EL one-form is corresponding to the EL equations and it is a special case for coboundary EL one-forms

$$E(u^i, u^j_\mu) = \mathrm{d}\alpha(u^i, u^j_\mu)\,, \tag{19}$$

where $\alpha(u^i, u^j_\mu)$ is an arbitrary smooth function of (u^i, u^j_μ). Although they are cohomologically trivial but it is already seen that in the EL one-forms, (u^i, u^j_μ) are ***not*** in the solution space of the EL equations only rather they are in the function space in general.

We now take the exterior derivative d of the Lagrangian. In terms of the EL one-form we get

$$\mathrm{d}L(u^i, u^j_\mu) = E(u^i, u^j_\mu) + \partial_\mu\theta^\mu\,, \tag{20}$$

where θ^μ are the canonical one-forms

$$\theta^\mu = \frac{\partial L}{\partial u^i_\mu}\mathrm{d}u^i\,. \tag{21}$$

From Eq. (20), it is easy to see that the EL one-forms are not coboundary in general since the canonical one-forms θ^μ are not coboundary so that the EL cohomology is not trivial in general.

By virtue of the nilpotency of d, $\mathrm{d}^2L(u^i, u^j_\mu) = 0$, it is easy to prove that if and only if the EL condition is satisfied, i.e., the EL one-form is closed

$$\mathrm{d}E(u^i, u^j_\mu) = 0\,, \tag{22}$$

the multisymplectic structure preserving (MSSP) property, i.e., the multisymplectic conservation or divergence free law follows

$$\sum_{\mu=1}^{n}\frac{\partial}{\partial x^\mu}\omega^\mu = 0\,, \tag{23}$$

where the symplectic structures ω^μ are given by

$$\omega^\mu = \mathrm{d}\theta^\mu = \frac{\partial^2 L}{\partial u^i_\mu \partial u^j}\mathrm{d}u^i \wedge \mathrm{d}u^j + \frac{\partial^2 L}{\partial u^i_\mu \partial u^j_\mu}\mathrm{d}u^i \wedge \mathrm{d}u^j_\mu\,. \tag{24}$$

Similar to the finite dimensional case, it is also important to note again that although the null EL one-form, the coboundary EL one-forms satisfy the EL condition, it does not mean that the closed EL one-forms can always be exact as pointed out above. In addition, $u^i(x)$s in the EL condition are ***not*** in the solution space of the EL equations only in general. Therefore, the MSSP law holds, i.e., the multisymplectic two-forms ω^μ are conserved, not only in the solution space of the equations but also in the function space with the closed EL condition in general.

3 The Discrete Symplectic and Multisymplectic Structures in Lagrangian Formalism

Now we consider certain difference discrete versions of the symplectic and multisymplectic structures and their preserving properties in the Lagrangian mechanism and field theory studied in the last section.

3.1 *The Discrete Symplectic Structure in Discrete Lagrangian Mechanism*

Let us first consider the symplectic structure and its preserving in the Lagrangian mechanism in case that "time" t is discretized while the configuration space at each moment of t is still continuous.

Let us assume, without loss any generality, that in the course of numerical calculation, the "time" $t \in R$ is discretized with equal spacing $h = \Delta t$,

$$t \in R \to t \in T_D = \{(t_k, t_{k+1} = t_k + h, \quad k \in Z)\}\,. \tag{25}$$

At the moment t_k, the configuration space is $M_k^n \in M_{T_D}^n = \{\cdots M_1^n \times \cdots \times M_k^n \cdots\}$, its coordinates are denoted by $q^i_{(k)}$.

The difference discrete Lagrangian can be written by

$$L_{D(k)} = L_D(q^i_{(k)}, q^i_{t(k)})\,, \tag{26}$$

where $q^j_{t\,(k)}$ is (forward) difference of $q^j{}_{(k)}$ at t_k defined by

$$\Delta_t q^j_{(k)} := \frac{\partial}{\partial t} q^j_{(k)} = q^j_{t(k)} = \frac{1}{h}\{q^j{}_{(k+1)} - q^j{}_{(k)}\}\,. \tag{27}$$

It is the (discrete) derivative on $T(T_D)$ in the sense of NCDC of a regular lattice L^1 with equal spacing[5] and the same notation for it as in the continuous case has been employed. It should be noted that in what follows the difference is always viewed as an entire geometric object and its dual $\mathrm{d}t$ is the base of $T^*(T_D)$.

We now consider the DDVP of the action functional

$$S_D = \sum_{k \in Z} L_D(q^i_{(k)}, q^j_{t(k)})\,,$$
$$\delta S_D = \sum_{k \in Z}\Big\{\frac{\partial L_{D(k)}}{\partial q^i_{(k)}}\delta q^i_{(k)} + \frac{\partial L_{D(k)}}{\partial q^i_{t(k)}}\delta q^i_{t(k)}\Big\}\,. \tag{28}$$

By means of the modified Leibniz law with respect to Δ_t[5,6] (see the Appendix), we have

$$\Delta_t\Big(\frac{\partial L_{D(k-1)}}{\partial q^i_{t(k-1)}}\delta q^i_{(k)}\Big) = \frac{\partial L_{D(k)}}{\partial q^i_{(k)}}\delta q^i_{t(k)} + \Delta_t\Big(\frac{\partial L_{D(k-1)}}{\partial q^i_{t(k-1)}}\Big)\delta q^i_{(k)}\,.$$

Therefore,

$$\delta S_D = \sum_{k \in Z}\Big\{\frac{\partial L_{D(k)}}{\partial q^i_{(k)}} - \Delta_t\Big(\frac{\partial L_{D(k-1)}}{\partial q^i_{t(k-1)}}\Big)\Big\}^i_{(k)} + \sum_{k \in Z}\Delta_t\Big(\frac{\partial L_{D(k-1)}}{\partial q^i_{t(k-1)}}\delta q^i_{(k)}\Big)\,.$$

Using the properties

$$\sum_{k \in Z}\Delta_t f(t_k) = f(t_k)\,\Big|_{k=-\infty}^{k=+\infty}\,, \tag{29}$$

and assuming δq^i_k satisfy

$$\delta q^i(\pm\infty) = 0\,,$$

it follows the DEL equations

$$\frac{\partial L_{D(k)}}{\partial q^i_{(k)}} - \Delta_t\Big(\frac{\partial L_{D(k-1)}}{\partial q^i_{t(k-1)}}\Big) = 0\,. \tag{30}$$

It is easy to see that for the case with difference discrete Lagrangian

$$L_D^{(k)}(q^{i\,(k)}, \Delta_t q^{j\,(k)}) = \frac{1}{2}(\Delta_t q^{i\,(k)})^2 - V(q^{i\,(k)})\,, \tag{31}$$

the DDVP gives the DEL equations

$$\Delta_t(\Delta_t q^{i\,(k-1)}) - V'(q^{i\,(k)}) = 0\,,$$

i.e.,

$$\frac{1}{h^2}(q^{i\,(k+1)} - 2q^{i\,(k)} + q^{i\,(k-1)}) = V'(q^{i\,(k)})\,. \tag{32}$$

This is just what is wanted for the difference discrete counterpart of the equations in the continuous case.

Now we consider the difference discrete symplectic structure and its preserving property. Taking the exterior derivative d on $L_{D(k)}$, we get

$$\mathrm{d}L_{D(k)} = \frac{\partial L_{D(k)}}{\partial q^i_{(k)}}\mathrm{d}q^i_{(k)} + \frac{\partial L_{D(k)}}{\partial q^i_{t(k)}}\mathrm{d}q^i_{t(k)}\,.$$

By means of the modified Leibniz law with respect to Δ_t and introducing the DEL one-form

$$E_{D(k)}(q^i_{(k)}, q^j_{t(k)}) := \Big\{\frac{\partial L_{D(k)}}{\partial q^i_{(k)}} - \Delta_t\Big(\frac{\partial L_{D(k-1)}}{\partial q^i_{t(k-1)}}\Big)\Big\}\mathrm{d}q^i_{(k)}\,, \tag{33}$$

we have

$$\mathrm{d}L_{D(k)} = E_{D(k)} + \Delta_t\theta_{D(k)}\,, \tag{34}$$

where $\theta_{D(k)}$ is the discrete canonical one-form

$$\theta_{D(k)} = \frac{\partial L_{D(k-1)}}{\partial q^i_{t(k-1)}}\mathrm{d}q^i_{(k)}\,, \tag{35}$$

and there exists the following discrete symplectic two-form on $T^*(M^n_{T_D})$,

$$\omega_{D(k)} = \mathrm{d}\theta_{D(k)} = \frac{\partial^2 L_{D(k-1)}}{\partial q^i_{t(k-1)}\partial q^j_{(k-1)}}\mathrm{d}q^j_{(k-1)} \wedge \mathrm{d}q^i_{(k)} + \frac{\partial^2 L_{D(k-1)}}{\partial q^i_{t(k-1)}\partial q^j_{t(k-1)}}\mathrm{d}q^j_{t(k-1)} \wedge \mathrm{d}q^i_{(k)}\,. \tag{36}$$

Now by virtue of the nilpotency of d on $T^*(M^n_{T_D})$, we get

$$0 = \mathrm{d}^2 L_{D(k)} = \mathrm{d}E_{D(k)} + \Delta_t\omega_{D(k)}\,. \tag{37}$$

Therefore, it is easy to see that if and only if the DEL one-form satisfies what is called the DEL condition, i.e., it is closed

$$\mathrm{d}E_{D(k)} = 0\,, \tag{38}$$

then it gives rise to the discrete (difference) symplectic structure-preserving law,

$$0 = \Delta_t\omega_{D(k)} \tag{39}$$

Similar to the continuous case, the null DEL one-form, which is corresponding to the DEL equation, is a special case of the coboundary DEL one-forms,

$$E_{D(k)} = \mathrm{d}\alpha_{D(k)}(q^i_{(k)}, q^j_{t(k)})\,, \tag{40}$$

where $\alpha_{D(k)}(q^i_{(k)}, q^j_{t(k)})$ is an arbitrary function of $(q^i_{(k)}, q^j_{t(k)})$. Although they satisfy the DEL condition, this does not mean that all closed DEL one-forms are exact. In fact equation (34) shows that the EL one-forms are not exact in general since the canonical one-form $\theta_{D(k)}$ is not trivial. In addition, $(q^i_{(k)}, q^j_{t(k)})$ are ***not*** in the solution space of the DEL equations only rather they are still in the function space with the DEL condition in general. Therefore, this also means that the DDSSP law holds in the function space with the DEL cohomology in general rather than on the solution space only.

3.2 *The Discrete Multisymplectic Structure in Discrete Lagrangian Field Theory*

We now study the discrete multisymplectic structure in discrete Lagrangian field theory. For the sake of simplicity, let us consider the $1+1-d$ and $2-d$ cases in discrete Lagrangian field theory (DLFT) for a scalar field. Let X^2 with suitable signature of the metrics be the base manifold, L^2 a regular lattice with two-directions x_μ, $\mu = 1, 2$ on X^2, M_D the discrete configuration space with $u^{(i,j)} \in M_D$.

The difference discrete Lagrangian is denoted as

$$L_D^{(i,j)} = L_D(u^{(i,j)}, u_\mu^{(i,j)})\,, \tag{41}$$

where

$$\Delta_1 u^{(i,j)} = \frac{1}{h}(u^{(i+1,j)} - u^{(i,j)})\,,$$
$$\Delta_2 u^{(i,j)} = \frac{1}{h}(u^{(i,j+1)} - u^{(i,j)})\,.$$

They are the bases of $T(M_D)$ and their duals $\mathrm{d}x^\mu = \mathrm{d}_L x^\mu$ are the bases of $T^*(M_D)$,

$$\mathrm{d}_L x^\mu(\partial_\nu) = \delta^\mu_\nu\,.$$

The action functional is given by

$$S_D = \sum_{\{i,j\}\in Z\times Z} L_D(u^{(i,j)}, u_\mu^{(i,j)})\,. \tag{42}$$

Taking the variation of S_D and regarding the differences as the entire geometric objects, we get

$$\delta S_D = \sum_{\{i,j\}\in Z\times Z} \Big\{ \frac{\partial L_D^{(i,j)}}{\partial u^{(i,j)}} \delta u^{(i,j)} + \frac{\partial L_D^{(i,j)}}{\partial u_\mu^{(i,j)}} \delta u_\mu^{(i,j)} \Big\}\,.$$

Employing the modified Leibniz law, we have

$$\Delta_1\Big(\frac{\partial L_D^{(i-1,j)}}{\partial u_1^{(k-1,l)}}\delta u^{(k,l)}\Big) = \frac{\partial L_D^{(i,j)}}{\partial u_1^{(k,l)}}\delta u_1^{(k,l)} + \Delta_1\Big(\frac{\partial L_D^{(i-1,j)}}{\partial u_1^{(k-1,l)}}\Big)\delta u^{(k,l)}\,,$$
$$\Delta_2\Big(\frac{\partial L_D^{(i,j-1)}}{\partial u_2^{(k,l-1)}}\delta u^{(k,l)}\Big) = \frac{\partial L_D^{(i,j)}}{\partial u_2^{(k,l)}}\delta u_2^{(k,l)} + \Delta_2\Big(\frac{\partial L_D^{(i,j-1)}}{\partial u_2^{(k,l-1)}}\Big)\delta u^{(k,l)}\,.$$

Assuming that $\delta u^{(k,l)}$s vanish at infinity, it follows the DEL equations

$$\frac{\partial L_D^{(i,j)}}{\partial u^{(k,l)}} - \Delta_1\Big(\frac{\partial L_D^{(i-1,j)}}{\partial u_1^{(k-1,l)}}\Big) - \Delta_2\Big(\frac{\partial L_D^{(i,j-1)}}{\partial u_2^{(k,l-1)}}\Big) = 0\,. \tag{43}$$

Let us consider an example with the discrete Lagrangian

$$L_D^{(i,j)}(u^{(i,j)}, u_\mu^{(i,j)}) = \frac{1}{2}(\Delta_\mu u^{(i,j)})^2 - V(u^{(i,j)})\,. \tag{44}$$

The DDVP gives the DEL equations

$$\Delta_1(\Delta_1 u^{(i-1,j)}) + \Delta_2(\Delta_2 u^{(i,j-1)}) - V'(u^{(i,j)}) = 0\,,$$

i.e.,

$$\frac{1}{h_1^2}(u^{(i+1,j)} - 2u^{(i,j)} + u^{(i-1,j)}) + \frac{1}{h_2^2}(u^{(i,j+1)} - 2u^{(i,j)} + u^{(i,j-1)}) = V'(u^{(i,j)})\,. \tag{45}$$

This is also what is wanted for the difference discrete coun-

We now consider the multisymplectic properties of the DLFT. Taking exterior derivative $\mathrm{d} \in T^*(M_D)$ of $L_D^{(i,j)}$ and making use of the modified Leibniz law, we get

$$\mathrm{d}L_D^{(i,j)} = E_D^{(i,j)}(u^{(i,j)}, u_\mu^{(i,j)}) + \Delta_\mu \theta^{\mu(i,j)}\,, \tag{46}$$

where $E_D^{(i,j)}$ is the DEL one-form defined by

$$E_D^{(i,j)}(u^{(i,j)}, u_\mu^{(i,j)}) := \Big\{\frac{\partial L_D^{(i,j)}}{\partial u^{(k,l)}} - \Delta_1\Big(\frac{\partial L_D^{(i-1,j)}}{\partial u_1^{(k-1,l)}}\Big) - \Delta_2\Big(\frac{\partial L_D^{(i,j-1)}}{\partial u_2^{(k,l-1)}}\Big)\Big\}\mathrm{d}u^{(k,l)}\,, \tag{47}$$

and $\theta^{\mu(i,j)}$ are two Cartan one-forms,

$$\theta^{1(i,j)} = \frac{\partial L_D^{(i-1,j)}}{\partial u_1^{(k-1,l)}}\mathrm{d}u^{(k,l)}\,,$$
$$\theta^{2(i,j)} = \frac{\partial L_D^{(i,j-1)}}{\partial u_2^{(k,l-1)}}\mathrm{d}u^{(k,l)}\,. \tag{48}$$

It is easy to see that there exist two symplectic two-forms on $T^*(M_D)$,

$$\omega^{\mu(i,j)} = \mathrm{d}\theta^{\mu(i,j)}, \qquad \mu = 1, 2\,. \tag{49}$$

The equation $\mathrm{d}^2 L_D^{(i,j)} = 0$ on $T^*(M_D)$ leads to the conservation law or the divergence free equation of $\omega^{\mu(i,j)}$,

$$\Delta_\mu \omega^{\mu(i,j)} = 0\,, \tag{50}$$

if and only if the DEL one-form satisfies the DEL condition, i.e., it is closed

$$\mathrm{d}E_D^{(i,j)} = 0\,. \tag{51}$$

Similar to the continuous case, the null DEL one-form is corresponding to the DEL equations and it is a special case of coboundary DEL one-forms

$$E_D^{(i,j)} = \mathrm{d}\alpha_D^{(i,j)}\,, \tag{52}$$

where $\alpha_D^{(i,j)}$ is an arbitrary function on T^*M_D. Although they satisfy the DEL condition, it does not mean that all closed DEL one-forms are exact. As a matter of fact, from Eq. (46) it is easy to see that the EL one-forms are not exact in general since the two canonical one-forms $\theta^{\mu(i,j)}$ ($\mu = 1, 2$) are not trivial. In addition, this indicates that the variables $u^{(k,l)}$s are still in the function space in general rather than the ones in the solution space only. Consequently, this also means that the difference discrete multisymplectic structure-preserving law holds in the function space with the closed DEL condition in general rather than in the solution space only.

It should be pointed out that the scenario of the approach can be straightforwardly generalized to higher-dimensional cases of $X^{1,n-1}$ and X^n.

4 Remarks

A few remarks are in order.

1) The approach presented in this paper, which may call the EL cohomology approach, to the symplectic and multisymplectic geometry and their difference discrete versions in the Lagrangian formalisms is more or less different from other approaches.[3,4] The EL and the DEL cohomological concepts and relevant content such as the EL and DEL one-forms, the null EL and DEL one-forms, the coboundary EL and DEL one-forms as well as the EL and the DEL conditions have been introduced and they have played very crucial roles in each case to show that the symplectic and multisymplectic preserving properties and their difference discrete versions are in the function space with the closed EL/DEL condition in general rather than in the solution space only.

It has been mentioned that the EL and DEL cohomology in relevant case is not trivial and it is very closely related to the symplectic and multisymplectic structures as

and the role of the EL cohomology in each case should be further studied and some issues are under investigation.[8]

2) The difference discrete variational formalism presented here is also different from the one by Veselov.[9,10] We have emphasized that the difference as discrete derivative is an entire geometric object. This is obvious and natural from the view point of NCDC. The continuous limits of the results given here are correct as well.

3) The NCDC on the regular lattices are employed in our approach. Since the differences do not satisfy the ordinary Leibniz law, in order to study the symplectic and multisymplectic geometry in these difference discrete systems it is natural and meaningful to make use of the NCDC.

4) The approach presented here can be generalized to the case that the configuration space is also discretized. This is closely related to the case of difference discrete phase space approach to the finite dimensional systems with separable Hamiltonian.[5,6]

5) It should be mentioned that the approach with the EL cohomological concepts can also directly be applied to the PDEs.

Let us for example consider the following type of equations[3,11] and make use of the same notations in Refs [3] and [11],

$$Kz_{x_1} + Lz_{x_2} = \nabla_z S(z)\,. \tag{53}$$

Introducing the EL one-form

$$E(z, z_{x_1}, z_{x_2}) := \mathrm{d}z^T\{Kz_{x_1} + Lz_{x_2} - \nabla_z S(z)\}\,, \tag{54}$$

it is easy to see that the null EL one-form gives rise to Eq. (54) and it is a special case of the coboundary EL one-forms,

$$E(z, z_{x_1}, z_{x_2}) = \mathrm{d}\alpha(z, z_{x_1}, z_{x_2})\,, \tag{55}$$

where $\alpha(z, z_{x_1}, z_{x_2})$ is an arbitrary function of (z, z_{x_1}, z_{x_2}).

Now by taking the exterior derivative d of the EL one-form, it is straightforward to prove that

$$\mathrm{d}E(z, z_{x_1}, z_{x_2}) = \frac{1}{2}\partial_{x1}(\mathrm{d}z^T \wedge K\mathrm{d}z) + \frac{1}{2}\partial_{x_2}(\mathrm{d}z^T \wedge L\mathrm{d}z).$$

This means that the following MSSP equation

$$\partial_{x_1}\omega + \partial_{x_2}\tau = 0 \tag{56}$$

holds, where

$$\omega = \mathrm{d}z^T \wedge K\mathrm{d}z, \qquad \tau = \mathrm{d}z^T \wedge L\mathrm{d}z\,,$$

if and only if the EL one-form is closed, i.e.,

$$\mathrm{d}E(z, z_{x_1}, z_{x_2}) = 0\,. \tag{57}$$

It should be mentioned that first from the definition of the EL one-form, it is not trivial in general since the first two terms in the definition are the canonical one-forms which are not trivial so that the EL cohomology is not

derived here is not dependent on the type of equations but can be applied to the equations so that it holds not only in the solution space of the equations but also in the function space relevant to the cohomology in general as well.

6) In principle, the cohomological scenario presented here should be available to not only to PDEs but also to ODEs and numerical schemes as well.

7) Finally, it should be pointed out that there exist lots of other problems to be studied.

Appendix

In this appendix we briefly recall some content of NCDC on lattice.[5−7]

A.1 An NCDC on an Abelian Discrete Group

Let G be an Abelian discrete group with a generator t, A the algebra of complex valued functions on G.

The left and right multiplications of a generator of G on its element are commute to each other since G is Abelian. Let us introduce right action on A that is given by

$$R_t f(a) = f(a\cdot t)\,, \tag{A1}$$

where $f \in A$, $a \in G$, t the generator and "$\cdot$" the group multiplication.

Let V be the space of vector fields,

$$V = \mathrm{span}\{\partial_t\}\,,$$

where ∂_t is the derivative with respect to the generator t given by

$$(\partial_t f)(a) \equiv R_t f(a) - f(a) = f(a\cdot t) - f(a)\,. \tag{A2}$$

The dual space of V, the space of one-form, is $\Omega^1 = \mathrm{span}\{\chi^t\}$ that is dual to V,

$$\chi^t(\partial_t) = 1\,. \tag{A3}$$

The whole differential algebra Ω^* can also be defined as $\Omega^* = \bigoplus_{n=0,1}\Omega^n$ with $A = \Omega^0$.

Let us define the exterior differentiation in

$$\Omega^*\mathrm{d}:\ \Omega^0 \to \Omega^1\,.$$

It acts on a zero-form $\omega^0 = A$ is as follows:

$$\mathrm{d}f = \partial_t f \chi^t \in \Omega^1\,. \tag{A4}$$

Now, the following theorem can straightforwardly be proved.

Theorem The exterior differentiation d satisfies

(a) $(\mathrm{d}f)(v) = v(f), \quad v\in V, \quad f\in\Omega^0\,,$

(b) $\mathrm{d}^2 = 0\,,$

if and only if

$$\text{(i)}\quad \mathrm{d}\chi^t = 0\,,$$
$$\text{(ii)}\quad \chi^t f = (R_t f)\chi^t\,. \tag{A6}$$

As was shown here, in order to establish a well-defined differential algebra, it is necessary and sufficient to introduce the noncommutative property of the multiplication between function and one-form.

The conjugation * on the whole differential algebra Ω^* and metric on discrete Abelian group can also be defined.

A.2 An NCDC on Regular Lattice

Let us consider the discrete translation group $G^m = \otimes_{i=1}^m G^i$, A the function space on G^m and a regular lattice with equal spacing L^m on an m-dimensional space R^m. Here G^i the i-th discrete translation group with one generator acting on one-dimensional space with coordinate q in such a way

$$R_{q^i}: q^i_n \to q^i_{n+1} = q^i_n + h\,, \qquad h\in R_+\,, \tag{A7}$$

R_{q^i} the discrete translation operation of the group G^i and it maps q^i_n of n-th size of q^i to the one q^i_{n+1} at $(n+1)$-th size, h the discrete translation step-length and R_+ the positive real number. It is easy to see that the action of G^i on i-th one-dimensional space R^1 generates the i-th chain L^i, $i = 1,\ldots,m$, with equal spacing h. Similarly, the regular lattice L^m with equal spacing h is generated by G^m acting on R^m. Since there is a one-to-one correspondence between sizes on L^i and elements of G^i, one may simply regard L^i as G^i. For the same reason, one may simply regard L^m as G^m.

On the sizes of the regular lattice L^m, there are discrete coordinates q^i_n, $i = 1,\ldots,m$. There is a set of generators in the discrete translation group G^m acting on L^m in such a way

$$R_{q^i}:\ q^i_n \to q^i_{n+1}\,, \qquad i = 1,\ldots,m\,. \tag{A8}$$

With respect to the generators there is a set of independent derivatives ∂_{q^i} on $f_n(q^i) = f(q^i_n)\in A$. They should be defined as the correspondent differences of the functions valued at two nearest sizes, i.e.,

$$\begin{aligned}\partial_{q^i} f(q^i_n) = \Delta_{q^i} f(q^i_n) &= \frac{1}{h}[(R_{q^i} - i\,\mathrm{d})f(q^i_n)]\\ &= \frac{1}{h}[f(q^i_{n+1}) - f(q^i_n)]\,.\end{aligned} \tag{A9}$$

The differential one-form is defined by

$$\mathrm{d}f = \partial_{q^i} f\,\mathrm{d}q^i = \Delta_{q^i} f\,\mathrm{d}q^i\,, \qquad f\in A\,. \tag{A10}$$

The two-forms and the whole differential algebra Ω^* can also be defined. Here d is the exterior differentiation. Similarly, the following theorem can be proved for d.

i.e.,

$$\begin{aligned} &\mathrm{d}^2 = 0\,, \\ &\mathrm{d}(\omega \wedge \omega') = \mathrm{d}\omega \wedge \omega' \\ &\qquad + (-1)^{\deg \omega} \omega \wedge \mathrm{d}\omega', \quad \omega, \omega' \in \Omega^*\,, \end{aligned} \tag{A11}$$

if and only if

$$f(q^i + h)\mathrm{d}q^i = \mathrm{d}q^i f(q^i)\,. \tag{A12}$$

This gives

$$q^i \mathrm{d}q^i - \mathrm{d}q^i q^i = -h \mathrm{d}q^i\,.$$

The above two equations show the noncommutative properties between the functions (including the coordinates) and differential forms.

From these properties, it follows the modified Leibniz rule for derivatives,

$$\Delta_{q^i}(fg) = \Delta_{q^i} f \cdot g + \{R_{q^i} f\} \cdot \Delta_{q^i} g\,. \tag{A13}$$

It should be noted that the definitions and relations given above for the NCDC on the regular lattice L^m are at least formally very similar to the ones in the ordinary commutative differential calculus (CDC) on R^m. The differences between the two cases are commutative or not.

The Hodge $*$ operator and the co-differentiation operator

$$\delta_L :\ \Omega^k \to \Omega^{k-1}$$

on the regular lattice L^m can also be defined similarly to the ones on R^m. Consequently, the Laplacian on the lattice L^m may also be given by

$$\Delta_L = \mathrm{d}\delta_L + \delta_L \mathrm{d}\,. \tag{A14}$$

It is in fact the discrete counterpart of the Laplacian Δ on R^m. For other objects and/or properties on R^m, there may have the discrete counterparts on L^m as well. For example, the null-divergence equation of a form ω on R^m reads

$$\delta\alpha = 0\,. \tag{A15}$$

Its counterpart on the lattice L^m is simply

$$\delta_L \alpha_L = 0\,. \tag{A16}$$

This is the forward difference form of null-divergence equation.

In the case of $L^{1,m} \in R^{1,m}$ with Lorentz signature, these equations become the conservation law of α and its difference form of α_L. This is available not only for the symplectic geometry and symplectic algorithms but also the multisymplectic geometry and multisymplectic algorithms as well. It should be emphasized that for the discrete counterparts on the lattice, they obey the NCDC on the lattice L^m rather than the CDC on R^m. This is the most important point.

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